Rapport de stage

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Contents

Abstract

The main goal of this study was to introduce all the mathematical tools starting from the notion of manifold to the Einstein field equation from the General Relativity. Thus a generalization of tools such as derivatives, curves and distance had to be done in order to highlights invariants such as curvature that can be helpful to classify spaces. The study will explore some properties along the way for a better understanding of the different layers of mathematical objects.

1 Smooth Manifold

Definition 1.1 (topological n-manifold)**.** *If E is an Hausdorff space, second countable and locally Euclidean of dimension n, then E is a topological n-manifold.*

Definition 1.2 (Coordinate Chart)**.** *Let U be an open subset of a manifold M,* $\varphi : U \to \tilde{U} \subset \mathbb{R}^n$ *where* φ *is an homeomorphism. (U,* φ*) is called the Coordinate chart.*

These objects will be used in the case of a locally Euclidean spaces, thus by definition every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}$

Idea behind smooth n-manifold : $\varphi : U \to \tilde{U}$ and $f : U \to \mathbb{R}$ such that $f \circ \varphi^{-1} : \tilde{U} \subset \mathbb{R}^n \to \mathbb{R}$ could be smooth in the usual sense. But in order to be independent of the choice of the coordinate chart we say that (U, φ) and (V, ψ) are smoothly compatible if either : $U \cap V = \emptyset$

or the transition map from φ to ψ : $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Definition 1.3. *An atlas is a collection of charts whose domain covers M*

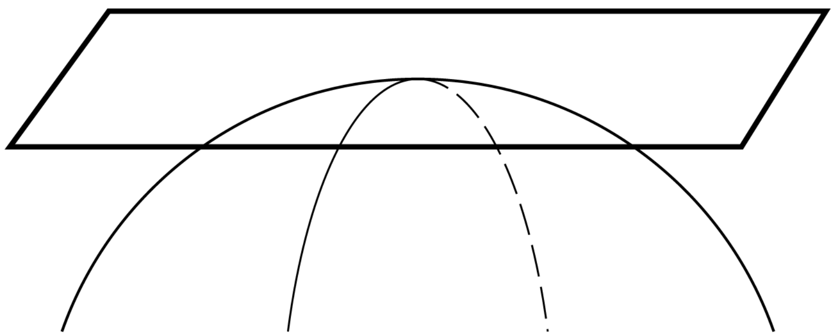
Definition 1.4. *A smooth atlas for a topological manifold M is an atlas for M such that each transition map is a smooth map, and two smooth atlases for M are smoothly equivalent provided their union is again a smooth atlas for M.*

Definition 1.5. (A, M) *is a smooth structure if* A *is a smooth maximal atlas and* M *a topological manifold. Where the smooth maximal atlas is found by taking the union of all atlases belonging to a smooth structure.*

Thus a ***smooth manifold*** *is a topological manifold M together with a smooth structure on M.*

Thus we can define a **smooth map** $F : M \to N$ as
for every p in M, be (U, φ) a smooth chart containing p and be (V, ψ) a smooth chart containing $F(p)$, $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$, both subset of $\mathbb{R}^n$.

2 Tangent Space



We need to define the tangent space without an ambiant Euclidean space.

Definition 2.1. *Let* M *be a smooth manifold,* p *a point of* M*,* X *a linear map :* $\mathcal{C}^\infty(M) \to \mathbb{R}$*.*

X *is called a derivative at* p *if* $\forall f, g \in \mathcal{C}^\infty(M), X(fg) = f(p)Xg + g(p)Xf$*.*
The set of all derivative of $\mathcal{C}^\infty(M)$ *at* p *is a vector space called the tangent space to* M *at* p*, denoted* T_pM *and an element of* T_pM *is called a tangent vector.*

2.1 Constructions in $\mathcal{C}^\infty(\mathbb{R}^n)$

Lemma 2.2. *Suppose* $a \in \mathbb{R}^n$ *and* $X \in T_a(\mathbb{R}^n)$

1. *if* f *is a constant function, then* $Xf = 0$

2. *if* $f(a) = g(a) = 0$*, then* $X(fg) = 0$

Proof.

1. Let $X \in T_a(\mathbb{R}^n)$, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\forall u \in \mathcal{V}_a(\mathbb{R}^n)$, $f(u) = 1$. Thus $X(f) = X(ff) = 2f(a)Xf = 2Xf \Rightarrow Xf = 0$. Then if $g(u) = c \Rightarrow g = cf$ so $X(g) = cX(f) = 0$.

2. $f(a) = g(a) = 0 \Rightarrow X(fg) = 0$ by the product rule.

$\square$

Proposition 2.3. *For any $a \in \mathbb{R}^n$ the map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}^n$ to $T_a(\mathbb{R}^n)$.*

Proof. $v = v^i e^i|_a$ in component near a. So if we take f as the j-th coordinate function $x^j : \mathbb{R}^n \to \mathbb{R}$ we have

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a) = v^i \frac{\partial}{\partial x^i}(x^j)|_{x=a} = v^j$$

but $D_v|_a f = 0$. Thus $\forall j, v^j = 0$ so $v = 0$ because v^j are its coordinates. To prove the surjectivity : Let X be derivation at a, we define $v^i = X(x^i)$ and we will show that $X = D_v|_a$ where $v = v^i e_i$.

Taylor formula at 1st order gives :

$$f(x) - f(a) = \sum_{i=1}^{n} (x^i - a^i) \frac{\partial f}{\partial x^i}(a) + \sum_{i=1}^{n} g_i(x)(x^i - a^i)$$

with $g_i(x)$ vanishing at $x = a$. Thus if we apply X and use the Lemma 2.2

$$Xf = X(f(a)) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) X(x^i) = v^i \frac{\partial f}{\partial x^i}(a)$$

which is in fact just $D_v|_a f$. $\square$

Proposition 2.4. *For any $a \in \mathbb{R}^n$, the n derivatives $\{\frac{\partial}{\partial x^1}|_a, ..., \frac{\partial}{\partial x^n}|_a\}$, defined by $\frac{\partial}{\partial x^i}|_a f = \frac{\partial f}{\partial x_1}(a)$, form a basis for $T_a(\mathbb{R}^n)$, which therefore has a dimension of n.*

Proof. $\frac{\partial}{\partial x^i}|_a = D_{e^i}|_a$ so because (e^i) is a basis for $\mathbb{R}^n$, Proposition 2.3 gives that $(\frac{\partial}{\partial x^i}|_a)$ is a basis for $T_a(\mathbb{R}^n)$. $\square$

2.2 Constructions in $\mathcal{C}^\infty(M)$ with M manifold.

Lemma 2.2 was proven without using $\mathbb{R}^n$

Definition 2.5. *Let φ be a smooth map between smooth manifolds M and N. We can define the pushforward of X by φ as $\forall f \in \mathcal{C}^\infty(M), \forall X \in T_pM$,*

$$\varphi_* X(f) = X(f \circ \varphi)$$

. Watch out that it is dependent of $p \in M$.

Where $\mathcal{C}^\infty(M)$ represent the set of all smooth real-valued functions and $\varphi_* : T_xM \to T_{\varphi(x)}M$ such that it can be identified with $d\varphi_x$, $d\varphi_x(X)(f) = X(f \circ \varphi)$, knowing that any tangent vector $X \in T_pM$ can be viewed as a derivation acting on smooth real-valued functions $\mathcal{C}^\infty(M)$.
This will be highlighted when we will introduce the coordinate notation.

Proof. $\varphi_* X$ is clearly linear and is a derivative at $\varphi(p)$ because $\forall f, g \in \mathcal{C}^\infty(M)$,

$$\varphi_* X(fg) = X(fg \circ \varphi) = X((f \circ \varphi)(g \circ \varphi)) = f \circ \varphi(p) X(g \circ \varphi) + g \circ \varphi(p) X(f \circ \varphi)$$

$$= f(\varphi(p))\varphi_* X(g) + g(\varphi(p))\varphi_* X(f).$$

□

Lemma 2.6. *Let $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$.*

1. *$F_* : T_pM \to T_{F(p)}N$ is linear*
2. *$(G \circ F)_* = G_* \circ F_* : T_pM \to T_{G \circ F(p)}P$*
3. *$(Id_M)_* = Id_{T_pM} : T_pM \to T_pM$*
4. *If F is a diffeomorphism, then $F_* : T_pM \to T_{F(p)}N$ is an isomorphism*

Proposition 2.7. *Suppose M is a smooth manifold, $p \in M$ and $X \in T_pM$. If f and g are smooth functions on M that agree on some neighborhood of p, then $Xf = Xg$.*

Proof. Let define $h = f - g$ and ψ a smooth bump function, that is equal to 1 whenever h is non zero and zero on the neighborhood.

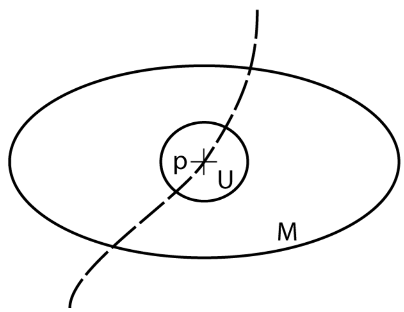
$$\psi(p) = h(p) = 0 \Rightarrow X(\psi h) = 0.$$

because of the generalization of Lemma 2.2. But by construction $\psi h = h$ so $Xf = Xg$. □

Using this proposition we can see that the tangent space for a submanifold is equal to the tangent space to the whole manifold.

Proposition 2.8. *Let M be a smooth manifold, let $U \subset M$ be an open submanifold and let $\iota : U \to M$ be the inclusion map. For any $p \in U$, $\iota_* : T_pU \to T_pM$ is an isomorphism.*

Proof. First proof : we can use the definition of tangent space by curves, because every curves of M going through p can be restricted as a curve in U and thus being in the same equivalence classes than the curves of U. Thus the set of curves in U describes the tangent space.

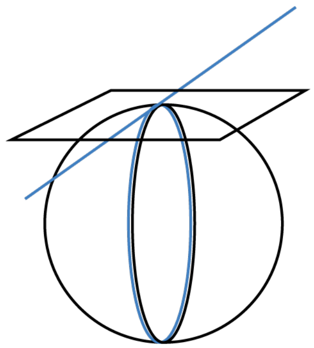


Second proof : Let f be a map in U and B a neighborhood of p such that $\overline{B} \subset U$. We define $\overline{f} \in \mathcal{C}^\infty(M)$, $\overline{f}|_{\overline{B}} = f$ by being f with a bump function with $U \backslash \overline{B}$ the smooth transition.
Let $X \in T_pU$ verify $\iota_* X = 0$.

Thus because they agree on some neighborhood, Proposition 2.7 gives $Xf = X(\overline{f}|_U) = X(\overline{f} \circ \iota) = \iota_* X\overline{f} = 0$. Because this hold for every $f \in \mathcal{C}^\infty(U)$, it follows that $X = 0$ and ι_* is injective.

Suppose $Y \in T_pM$ arbitrary. Let $X : \mathcal{C}^\infty(U) \to \mathbb{R}$ such that $Xf = Y\overline{f}$ where $\overline{f}$ is any function on all M that agree with f on $\overline{B}$. Xf is independent of the choice of $\overline{f}$, because of Proposition 2.7, so it is well defined.
We can check that $X \in T_pU$. For any $g \in \mathcal{C}^\infty(M)$, $(\iota_* X)g = X(g \circ \iota) = Y(\overline{g \circ \iota}) = Yg$, Thus ι_* is surjective. □

Remarks :



Th tangent space are distincts (line and plane) because the great circle is not an **open submanifold** of the sphere.

Proposition 2.9.

Let $\phi : U \to V$ be a smooth map from an open subset U of $\mathbb{R}^m$ to an open subset V of $\mathbb{R}^n$. For any point x in U, the Jacobian of ϕ at x (with respect to the standard coordinates) is the matrix representation of the total derivative of ϕ at x, which is a linear map $d\varphi_x : \mathbb{R}^m \to \mathbb{R}^n$.
We wish to generalize this to the case that φ is a smooth function between any smooth manifolds M and N.

Definition 2.10 (curves derivatives)**.** *Let M be a smooth manifold and γ be a curve $\mathbb{R} \to M$, $\gamma'(t_0) = \gamma_*(\frac{d}{dt}|_{t_0}) \in T_{\gamma(t_0)}M$ (because $\frac{d}{dt}|_{t_0}$) is a standard coordinate basis in $T_{t_0}\mathbb{R}$.*

3 Riemannien metrics

Definition 3.1 (Tensor)**.** *Let V be a finite dimensional vector space, we can define the covariant k-tensor as a multilinear map $T_k(V) : V \times ... \times V \to \mathbb{R}$, the contravariant k-tensor $T^k(V) : V^* \times ... \times V^* \to \mathbb{R}$ and the mixed $T^l_k(V) : V \times ... \times V \times V^* \times ... \times V^* \to \mathbb{R}$ with k copies of V and l copies of V^**

which can be respectively written $T_k(V) = V^* \otimes ... \otimes V^*$, $T^k(V) = V \otimes ... \otimes V$ and $T^l_k(V) = V^* \otimes ... \otimes V^* \otimes V \otimes ... \otimes V$.

Definition 3.2 (Bundle of tensors)**.** *Let M be smooth manifold. $T^l_k M = \coprod_{p \in M} T^l_k(T_pM)$ and $\mathcal{T}^l_k M = \{$smooth sections of $T^l_k M\}$.*

Definition 3.3. *Let M be a smooth manifold, a Riemannian metric g is a family of, smooth symmetric 2-tensor positive definite, g_p on the tangent space T_pM at each point p.*
(M,g) is called a Riemannian manifold, and g can be omitted if it is induced.

Definition 3.4. *The tangent bundle of M a smooth manifold, assigns for every p point of M a vector space, called the tangent vector space, T_pM.*

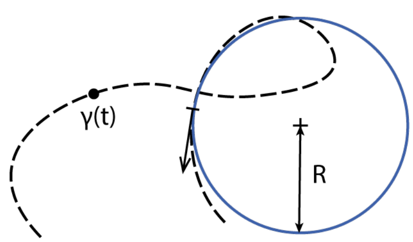
Definition 3.5. *Let $(x^1, ..., x^n) : U \to \mathbb{R}^n$ be a system of smooth local coordinates on M a smooth manifold. The vectors $\{\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p\}$ form a basis of the vector space T_pM, $\forall p$.*
Thus one can define the metric tensor "components" $\forall p$, $g_{ij}|_p = g_p(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p)$, which give that the matrix $((g_{ij}))$ is symmetric, definite, positive.
It can be written in terms of dual basis $(dx^1, ..., dx^n)$ of the cotangent bundle as $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$.
Finally if we introduce the Einstein notation and the symmetric product on covectors we can write $g = g_{ij}dx^i \otimes dx^j = g_{ij}dx^i dx^j$

Theorem 3.6. *Every smooth manifold admits a Riemannian metric*

Proof. Use of partitions of unity, theory not included in this study. □

4 Riemannian manifold

4.1 Curves



In the plane $\mathbb{R}^2$ formally we have $\kappa(t) = |\ddot{\gamma}(t)|$ to define curvature which in that case would be $\kappa(t) = \frac{1}{R}$ with R the radius of the osculating circle. Then choosing a normal vector field N along the curves we can consider that $\kappa(t)$ can be < 0.

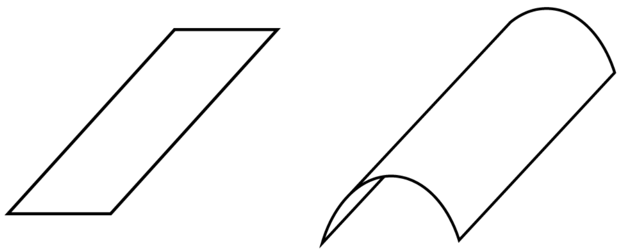
Theorem 4.1 (Plane curve classification theorem)**.** $\gamma \equiv \overline{\gamma}$ *(for N and $\overline{N}$)* $\Leftrightarrow \forall t, K_N(t) = K_{\overline{N}}(t)$.

Theorem 4.2 (total curvature theorem)**.** *If $\gamma : [a, b] \to \mathbb{R}^2$ is a unit speed closed curve such that $\dot{\gamma}(a) = \dot{\gamma}(b)$ and N is the inward pointing normal.*
Then $\int_a^b \kappa_N(t)dt = 2\pi$

Then we can define different parameters such as

- The principal curvatures of S at p : κ_1, κ_2, resp. minimum and maximum of the curvatures of all curves going through the point p on the surface S. They are not intrinsic.

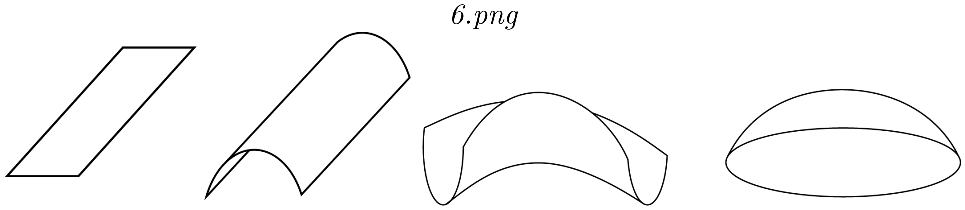
- The Gaussian curvature : $\kappa = \kappa_1 \kappa_2$ which is intrinsic.



On the left $\kappa_1 = \kappa_2 = 0 \Rightarrow \kappa = 0$ and on the right $\kappa_1 = 0, \kappa_2 = 1 \Rightarrow \kappa = 0$. But we have a diffeomorphism between these two surfaces.

Theorem 4.3 (uniformization theorem)**.** *Every connected 2-manifold is diffeomorphic to a quotient of one of the three constant curvature model surface such as :*

6.png



Resp. $\kappa = 0$ *;* $\kappa < 0$ *;* $\kappa > 0$

Therefore every connected 2-manifold has a complete Riemann metric with constant Gaussian curvature.

We will generalize this notion of curvature in a arbitrary manifold

4.2 Geodesics

Theorem 4.4. *A complete, connected Riemannian manifold M with constant sectionnal curvature is isometric to $\tilde{M}/\Gamma$ where $\tilde{M}$ is one of the constant curvature model spaces : $\mathbb{R}^n$, $S^n_{\mathbb{R}}$, $H^n_{\mathbb{R}}$ and Γ is a discrete group of isometries of $\tilde{M}$ isomorphic to $\pi_1(M)$, and acting freely and properly discontinuously on $\tilde{M}$.*

Theorem 4.5 (Cartan-Hadamard). *Suppose M is a complete, connected, Riemannian n-manifold with all sectionnal curvature less than or equal to zero Then the universal covering space of M is diffeomorphic to* $\mathbb{R}^n$.

Lemma 4.6. *Let V be a finite-dimensional vector space. There is a natural (basis independent) isomorphism between* $T^k_{l+1}(V)$ *and the space of multilinear maps :* $V^* \times ... \times V^* \times V \times ... \times V \to V$ *with l copies of* V^* *and k copies of V.*

We can define Trace (or contraction) which lowers the rank of a tensor by 2. $tr : T^1_1(V) \to \mathbb{R}$ is the usual trace. More generally : $tr : T^{k+1}_{l+1}(V) \to T^k_l(V)$ by $(trF)(\omega^1, ..., \omega^l, V_1, ..., V_k)$ being the trace of $F(\omega^1, ..., \omega^l, \bullet, V_1, ..., V_k, \bullet) \in T^1_1(V)$. In term of basis : $(trF)^{j_1 ... j_l}_{i_1 ... i_k} = F^{j_1 ... j_l m}_{i_1 ... i_k m}$.

From now on every manifold is taken as smooth, Hausdorff and second countable.

4.3 Vector bundles

Definition 4.7. *Let* $(E, M, \pi : E \to M)$ *be a smooth k-dimensional vector bundle. Which means that the total space E is a smooth manifold, the base space M is a smooth manifold and the projection* π *is a surjection map.*

1. *Each* $E_p = \pi^{-1}(p)$ *(called the* ***fiber*** *of E over p) is endowed by a structure pf a vector space*

2. *For each* $p \in M$*, there exists a neighborhood U of p and a diffeomorphism* $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ *(called a local trivialization of E) such that the following diagram commutes :*

Definition 4.8. *If* $\pi : E \to M$ *is a vector bundle over M, a* ***section*** *of E is a map* $F : M \to E$ *such that* $\pi \circ F = id$*, and a smooth section if the map F is smooth between manifolds.*

Lemma 4.9. *Let* $F : M \to E$ *be a section of a vector bundle.*
F is smooth if and only if the components $F^{j_1 ... j_l}_{i_1 ... i_k}$ *of F in term of any smooth coordinate map* $\{E_i\}$ *on an open set* $U \in M$ *depend smoothly on* $p \in U$

Definition 4.10. $\mathcal{T}(M)$ *is the space of smooth sections of TM (tangent bundle : bundle of all the tangent space* T_pM*). In other words* $\mathcal{T}(M)$ *is the space of smooth vector fields on M.*
In that perspective a tensor field on M is also a smooth section of some tensor bundle $T^k_l M$.

Definition 4.11. *A Riemannian metric g is a 2-tensor field (i.e. $g \in \mathcal{T}^2(M)$), symmetric, positive definite*

Definition 4.12 (Elementary Constructions)**.** *Raising and lowering indices : given a metric g on M, we can define a map, called flat, from TM to T^*M. Lowering index : $X \mapsto X^b$, such that $X^b(Y) = g(X, Y)$. We can write that*

$$X^b = g(X^i \partial_i, \bullet) = g_{ij} X^i dx^j$$

which can be written

$$X^b = X_j dx^j; X_j = g_{ij} X^i$$

Raising index : $\omega \mapsto \omega^\#$ being the inverse map. $\omega^i = g^{ij}\omega_j$ with $((g^{ij}))$ the matrix inverse of $((g_{ij} = g(e_i, e_j))$.

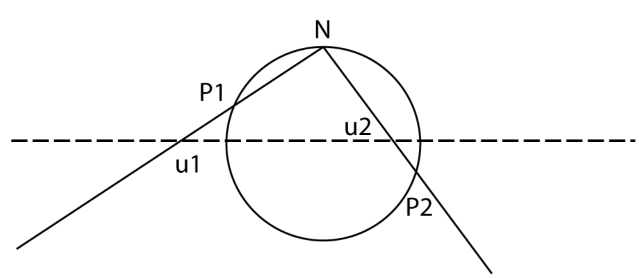
Definition 4.13 (Gradient)**.** *Let f be a real-valued, smooth function on a Riemannian manifold. We have $grad(f) = df^\# = g^{ij}\partial_i f \partial_j$.*

Definition 4.14.

- *Two metrics g_1 and g_2 on a manifold M are said to be conformal to each other if there is a positive definite function $f \in \mathcal{C}^\infty(M)$ such that $g_2 = f g_1$.*
- *Two Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$ are said to be conformally equivalent if there is a diffeomorphism $\varphi : M \to \tilde{M}$ such that $\varphi^* \tilde{g}$ is conformal to g.*

Remarks :

- We can define the trace of a 2-tensor on M a Riemannian manifold with $h^\#$ being a $\binom{1}{1}$-tensor thus we can define the $tr_g h = tr h^\# = g^{ij} h_{ij}$.
- Equivalence between $\mathbb{R}^n$ and $\mathcal{S}^n_R \subset \mathbb{R}^{n+1}$:
  The stereographic projection from north pole can be written $\sigma : \mathcal{S}^n_R \backslash N \to \mathbb{R}$ such that $P = (\zeta^1, ..., \zeta^n, \tau) \mapsto u \in \mathbb{R}^n$ where $U = (u^1, ..., u^n, 0)$ is the point where the line through N and P intersect the hyperplan $\{\tau = 0\}$ in $\mathbb{R}^{n+1}$.



Thus $\sigma(\zeta,\tau) = u = \frac{R}{R-\tau}\zeta$.

Lemma 4.15. *The stereographic projection is a conformal equivalence between* $\mathcal{S}^n_{\mathbb{R}}\backslash\{N\}$ *and* $\mathbb{R}^n$.

Proof. $\dot{g}_R$ being the metric on the sphere, $V \in T_q\mathbb{R}^n$,

$$(\sigma^{-1})^*\dot{g}_R(V,V) = \dot{g}_R(\sigma_*^{-1}V, \sigma_*^{-1}V) = \overline{g}(\sigma_*^{-1}V, \sigma_*^{-1}V)$$

where $\overline{g}$ is an Euclidien metric on $\mathbb{R}^{n+1}$.

$$\sigma_*^{-1}V = V^i\frac{\partial\zeta^j}{\partial u^i}\frac{\partial}{\partial\zeta^j} + V^i\frac{\partial\tau}{\partial u^i}\frac{\partial}{\partial\tau} = V\zeta^j\frac{\partial}{\partial\zeta^j} + V\tau\frac{\partial}{\partial\tau}$$

$$V\zeta^j = V(\frac{2R^2u^j}{|u|^2+R^2} = \frac{2R^2V^j}{|u|^2+R^2} - \frac{4R^2u^j\langle V,u\rangle}{(|u|^2+R^2)^2}$$

$$V\tau = V(R\frac{|u|^2-R^2}{|u|^2+R^2}) = \frac{2R\langle V,u\rangle}{|u|^2+R^2} - \frac{2R(|u|^2-R^2)\langle V,u\rangle}{(|u|^2+R^2)^2} = \frac{4R^3\langle V,u\rangle}{(|u|^2+R^2)^2}$$

And because $V(|u|^2) = 2\sum_k V^k u^k = 2\langle V,u\rangle$.

$$\overline{g}(\sigma_*^{-1}V, \sigma_*^{-1}V) = \sum_{j=1}^{n}(V\zeta^j)^2 + (V\tau)^2 = \frac{4R^4}{(|u|+R^2)^2}|V|^2$$

□

Lemma 4.16. *The sphere is locally conformally float i.e. each point has a neighborhood that conformally equivalent to an open set of* $\mathbb{R}^n$. *The stereographic projection gives such an equivalence (for north pole neighborhood, take the projection from the south pole)*

Definition 4.17.

- *Minkowski metric : Lorentz metric* m *on* $\mathbb{R}^{n+1}$ *that is written in terms of coordinates* $(\zeta^1,...,\zeta^n,\tau)$ *as* $m = (d\zeta^1)^2 + ... + (d\zeta^u)^2 - (d\tau)^2$
- *Lorentz metrics : pseudo-Riemannian metrics of index 1*

Definition 4.18. *A pseudo-Riemannian metric on a smooth manifold* M *: is a symmetric 2-tensor field* g *that is non degenerate at each point* p *in* M*. This means that the only vector orthogonal to everything is the zero vector. More formally ,* $\forall Y \in T_pM, g(X,Y) = 0 \Leftrightarrow X = 0$

If g *is a Riemannian metric* (g_{ij}) *is positive definite. But pseudo-R. metrics doesn't need to be positive. Given a p.R.m.* g *and a point* p *in* M*,* *by using a simple extension of Gram-Schmidt algorithm one can construct a basis $(E_1, ..., E_n)$ for T_pM in which*

$$g = -(\varphi^1)^2 - ... - (\varphi^r)^2 + (\varphi^{r+1})^2 + ... + (\varphi^n)^2 \quad (1)$$

where r is called the index.

Remark : Einstein Lorentz metric in $\mathbb{R}^4$: in absence of gravity the law of physics have the same form in any coordinate system in which the Minkowski metric has the expression 1. The differing physical characteristic of "space" and "time" arise from the fact that they are subspaces on which g is positive definite and negative definite, respectively.

The general theory of relativity includes gravitational effects by allowing the Lorentz metric to vary from point to point.

4.4 Hyperbolic space H^n_R

Definition 4.19. *Different definitions that are mutually isometric*

1. *Hyperbolic Model : H^n_R is the "upper sheet" $\{\tau = 0\}$ of the two sheeted hyperboloid in $\mathbb{R}^{n+1}$ defined in coordinates $(\zeta^1, ..., \zeta^n, \tau)$ by the equation $\tau^2 - |\zeta|^2 = R^2$ with the metric $h^1_R = \iota^* m$ where $\iota : H^n_R \to \mathbb{R}^{n+1}$.*

2. *Poincar Ball Model : B^n_R is the ball of radius R in $\mathbb{R}^n$ with the metric given in coordinates $(u^1, ..., u^n)$ by $h^2_R = 4R^4 \frac{(du^1)^2 + ... + (du^n)^2}{(R^2 - |u|^2)^2}$*

3. *Poincar Half-Space Model : U^n_R is the upper half space in $\mathbb{R}^n$ defined in coordinates $(x^1, ..., x^{n-1}, y)$ by $\{y > 0\}$ with the metric $h^3_R = R^2 \frac{(dx^1)^2 + ... + (dx^{n-1})^2 + dy^2}{y^2}$*

Remarks : B^n_R and U^n_R make it clear that the hyperbolic metric is locally flat. A Riemannian manifold is said to be flat if its Riemannian curvature tensor is null everywhere. Thus a (pseudo) Riemannian manifold is conformally flat if each point has a neighborhood that can be mapped to flat space by a conformal transformation (a map that locally preserves angles but not necessarily lengths).

5 Connections

Definition 5.1. *$\pi : E \to M$ be a vector bungle over a manifold M and let $\varepsilon(M)$ denote the space of smooth sections of E.*
A connection in E is a map $\nabla : \mathcal{T}(M) \times \varepsilon(M) \to \varepsilon(M)$ such that $X, Y \mapsto \nabla_X Y$ satisfying :

1. *$\nabla_X Y$ is linear over $\mathcal{C}^\infty(M)$ in X :*

$$\forall f, g \in \mathcal{C}^\infty(M), \nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + f\nabla_{X_2}Y$$

2. *$\nabla_X Y$ is linear over $\mathbb{R}$ in Y :*

$$\forall a, b \in \mathbb{R}, \nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2,$$

3. *∇ satisfies the following product rule :*

$$\forall f \in \mathcal{C}^\infty(M), \nabla_X(fY) = f\nabla_X Y + (Xf)Y$$

∇ is read "del" and $\nabla_X Y$ is called the covariant derivative of Y in the direction X.

Lemma 5.2. *∇ a connection in a bundle E ; $X \in \mathcal{T}(M)$; $Y \in \varepsilon(M)$; $p \in M$. Then $\nabla_X Y|_p$ depends only on the values of X and Y in an arbitrary neighborhood of p.*

Proof. We use the product rule and a smooth bump function such as we did for $D_v|_a$. □

Lemma 5.3. *$\nabla_X Y|_p$ depends only on the values of Y in a neighborhood of p and the value of X at p.*

Proof. By linearity it suffices to show that $\nabla_X Y|_p = 0$ whenever $X_p = 0$. We choose a coordinate neighborhood U of p, and write $X = X^i\partial_i$ in coordinate on U with $X^i(p) = 0$. Thus

$$\nabla_X Y = \nabla_{X^i\partial_i}Y = X^i\nabla_{\partial_i}Y \Rightarrow \nabla_X Y|_p = X^i(p)\nabla_{\partial_i}Y|_p = 0$$

allowed because of Lemma 5.2. □

Because of Lemma 5.3 we can write $\nabla_{X_p}Y$ in place of $\nabla_X Y|_p$.

Definition 5.4. *A linear connection on M is a connection in TM. i.e. $\mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$.*

Remark : Even if it looks like the def of a $\binom{2}{1}$-tensor field it is not a tensor field because it is not linear over $\mathcal{C}^\infty(M)$ on Y, but instead satisfy the product rule.

If we choose a local frame $\{E_i\}$ for TM of an open set $U \subset M$, we can write $\nabla_{E_i}E_j = \Gamma^k_{ij}E_k$ where Γ^k_{ij} is called the Christoffel symbols of ∇.

Lemma 5.5. *Let ∇ be a linear connection, and let $X, Y \in \mathcal{T}(U)$ be expressed in terms of local frame by $X = X^i E_i; Y = Y^j E_j$ Then*

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma_{ij}^k) E_k$$

Definition 5.6. *The Euclidean Connection on $\mathbb{R}^n$ is defined as $\overline{\nabla}_X Y = (XY^j)\partial_j$*

Lemma 5.7. *Suppose M is a manifold covered by a single coordinate chart. There is a one-to-one correspondance between linear connections on M and choices of n^3 smooth function $\{\Gamma_{ij}^k\}$ on M by the rule :*

$$\nabla_X Y = (X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k)\partial_k$$

Proposition 5.8. *Every manifold admits a linear connection.*

Proof. The notion of partition of unity is needed for the proof, but it was out of the scope of my study. □

Lemma 5.9. *Let ∇ be a linear connection on M. There is a unique connection in each tensor bundle $T_l^k M$, also denoted ∇, such that the following conditions are satisfied*

1. *On TM, ∇ agrees with the given connection.*
2. *On $T^0 M$, ∇ is given by ordinary differentiation of functions*

   $$\nabla_X f = Xf = X^i \partial_i f$$

3. *∇ obeys the following product rule with respect to tensor product :*

   $$\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes \nabla_X G$$

4. *∇ commutes with all contractions : if "tr" denotes the trace on any pair of indices, $\nabla_X(trY) = tr(\nabla_X Y)$.*

This connection satisfies the following additional properties :

1. *∇ obeys the following product rule with respect to the natural pairing (between covector field and vector field) :*

   $$\nabla_X \langle \omega, Y \rangle = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle$$

2. *For any $F \in \mathcal{T}^k_l(M)$, vector fields Y_i and 1-forms ω^j :*

$$(\nabla_X F)(\omega^1, ..., \omega^l, Y_1, ..., Y_k) = X(F(\omega^1, ..., \omega^l, Y_1, ..., Y_k)$$
$$- \sum_{j=1}^{l} F(\omega^1, ..., \nabla_X \omega^j, ..., \omega^l, Y_1, ..., Y_k)$$
$$- \sum_{i=1}^{l} F(\omega^1, ..., \omega^l, Y_1, ..., \nabla_X Y^i, ..., Y_k)$$

Lemma 5.10 (Construction of the total covariant derivative)**.** *If ∇ is a linear connection on M, and $F \in \mathcal{T}^k_l(M)$, the map :*

$$\nabla F : \mathcal{T}^1(M) \times ... \times \mathcal{T}^1(M) \times \mathcal{T}(M) \times ... \times \mathcal{T}(M) \to \mathcal{C}^\infty(M)$$

given by $\nabla F(\omega^1, ..., \omega^l, Y_1, ..., Y_k, X) = \nabla_X F(\omega^1, ..., \omega^l, Y_1, ..., Y_k)$ defines a $\binom{k+1}{l}$-tensor field.

Lemma 5.11 (Tensor characterization lemma)**.** *A map $\mathcal{T} : \mathcal{T}^1(M) \times ... \times \mathcal{T}^1(M) \times \mathcal{T}(M) \times ... \times \mathcal{T}(M) \to \mathcal{C}^\infty(M)$ is induced by a $\binom{k}{l}$-tensor field as above if and only if it is multilinear over $\mathcal{C}^\infty$.*

Similarly : $\mathcal{T} : \mathcal{T}^1(M) \times ... \times \mathcal{T}^1(M) \times \mathcal{T}(M) \times ... \times \mathcal{T}(M) \to \mathcal{T}(M)$ is induced by a $\binom{k}{l+1}$-tensor field if and only if it is multilinear over $\mathcal{C}^\infty$.

Definition 5.12. *$(\nabla_v \alpha)_p$ is defined to satisfy tensor contraction and product rule. That is, $(\nabla_v \alpha)_p$ is defined as the unique 1-form at p such that $\nabla_v \alpha)_p(u_p) = \nabla_v \alpha(u_p)) - \alpha(\nabla_v(u_p))$, $\forall u_p$ in a neighborhood of p.*

The covariant derivative of a covector field along a covector field is again a covector field.

Definition 5.13. *∇F is called the total covariant derivation of F.*

Let $u \in \mathcal{C}^\infty(M)$, then $\nabla u \in \mathcal{T}^1(M)$ is just the 1-form du, because both tensor have the same action on vectors : $\langle \nabla u, X \rangle = .\nabla_X u = Xu = X^i \partial_i u = \langle du, X \rangle$.

Remark :

- $u \in \mathcal{T}^0 M \Rightarrow \nabla u : \mathcal{T}(M) \to \mathcal{C}^\infty(M)$.
- $\nabla^2 u = \nabla(\nabla u))$ is called the covariant Hessian of u.
- We will separate the indices resulting from the differentiation from the preceding indices. If Y is a vector field written $Y = Y^i \partial_i$, The component of ∇Y are written ${Y^i}_{;j}$ so that $\nabla Y = {Y^i}_{;j}\ \partial_i \otimes dx^j$. Thus ${Y^i}_{;j} = \partial_j Y^i + Y^k \Gamma^i_{jk}$.

Lemma 5.14. *Let ∇ be a linear connection. The component of the total covariant derivative of $\binom{k}{l}$-tensor field F with respect to a coordinate system :*

$$F^{j_1 \ldots j_l}_{i_1 \ldots i_k\,;m} = \partial_m F^{j_1 \ldots j_l}_{i_1 \ldots i_k} + \sum_{s=1}^{l} F^{j_1 \ldots p \ldots j_l}_{i_1 \ldots i_k} \Gamma^{j_s}_{mp} - \sum_{s=1}^{k} F^{j_1 \ldots j_l}_{i_1 \ldots p \ldots i_k} \Gamma^{p}_{m i_s}$$

5.1 Vector fields along curves

Definition 5.15.

- *Curve : always taken smooth, parametrized curves.*
- *Curve segment : curve whose domain I is closed.*
- *Smoothness of γ on I, if I has an endpoint it means by definition that γ extends to a smooth curve on some open interval containing I.*
- *Let $\gamma : I \to M$ be a curve, the velocity $\dot\gamma(t)$ of γ is invariantly defined as the pushforward $\gamma_*(\frac{d}{dt})$:*

$$\dot\gamma(t) f = \frac{d}{dt} f \circ \gamma(t) \ ; \ \dot\gamma(t) = \dot\gamma^i(t)\partial_i$$

- *A vector field along a curve is a smooth map : $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$.*
- *$J(\gamma)$: space of vector fields along γ.*
- *$\dot\gamma(t) \in T_{\gamma(t)}M$ which is smooth proven by $\dot\gamma^i(t)\partial_i$ because γ is smooth thus coordinates are smooth.*
- *Suppose $\gamma : I \to M$ curve ; $\tilde V \in \mathcal{T}(M)$ vector field on M. For each $t \in I$, let*

$$V(t) = \tilde V_{\gamma(t)} \tag{2}$$

  . V is smooth because $\mathcal{T}(M)$ is the set of smooth sections $s : M \to E$ and by definition γ is smooth thus $V = s \circ \gamma : I \to TM$ is smooth.
- *A vector field along γ is said to be extendible if there exists a vector field $\tilde V$ on a neighborhood of $\gamma(I)$ that is related to V according to the identity 2.*
- *For exemple : If $\gamma(t_1) = \gamma(t_2)$ and $\dot\gamma(t_1) \neq \dot\gamma(t_2)$ Then $\dot\gamma$ is not extendible.*

Definition 5.16 (Covariant derivatives along curves). *Let ∇ be a linear connection on M, for each curve $\gamma : I \to M$, ∇ determines a unique operator :*

$$D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$$

satisfying :

1. *Linearity over $\mathbb{R}$*

$$D_t(aV + bW) = aD_tV + bD_tW, \ \forall a, b \in \mathbb{R}$$

2. *Product rule :*

$$D_t(fV) = \dot{f}V + fD_tV, \ \forall f \in \mathcal{C}^\infty(I)$$

3. *If V is extendible, then for any extension $\tilde{V}$ of V :*

$$D_tV(t) = \nabla_{\dot{\gamma}(t)}\tilde{V}$$

For any $V \in \mathcal{T}(\gamma)$, D_tV is called the covariant derivative of V along γ.

5.2 Geodesics

Let M be a manifold and γ a curve; ∇ a linear connection on M.

Definition 5.17.

- *The acceleration of γ is the vector field $D_t\dot{\gamma}$ along γ.*
- *A curve is called a geodesic with respect to ∇ if its acceleration is zero i.e. $D_t\dot{\gamma} = 0$.*

Theorem 5.18 (Existence ane Uniqueness of Geodesics). *Let M be a manifold with linear connection. For any $p \in M$, any $V \in T_pM$, any $t_0 \in \mathbb{R}$. There exist an open interval $I \subset \mathbb{R}$ countaining t_0 and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = V$. Any two such geodesics agree on their common domain.*

Proposition 5.19 (The geodesic equation). *$\gamma : I \to M$ is a geodesic $\Leftrightarrow$ $\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0$ with $\gamma(t) = (x^1(t), ..., x^n(t))$.*

It follows that we can define the maximal geodesic for any p and V noted γ_V, initial point p and velocity V ($p = \pi(V)$ because $V \in T_pM$ thus no need to specify).

Definition 5.20. *Let M be a manifold with a linear connection ∇, V a vector field along a curve γ.*

V is said to be parallel along γ with respect to ∇ if $D_t V = 0$.

A vector field is said to be parallel if it is parallel on every curve.

Thus geodesics can be characterized as a curve whose velocity vector is parallel along the curve.

We also have : V is parallel if and only if its total covariant derivative ∇V vanishes identically.

Theorem 5.21 (Parallel translation)**.** *$\gamma : I \to M$; $t_0 \in I$; $V_0 \in T_{\gamma(t_0)}M$ Then $\exists! V$ parallel vector field along γ such that $V(t_0) = V_0$.*

V is called the parallel translate of V_0 along γ.

Let define $P_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ by setting $P_{t_0 t_1} V_0 = V(t_1)$ where V is the parallel translate of V_0 along γ.

Proposition 5.22. *A connection "connects" nearby tangent spaces.*

Elements of proof. $D_t V(t) = \nabla_{\dot{\gamma}(t)} \tilde{V}$ where $V(t) = \tilde{V}_{\gamma(t)}$ and $D_t V(t_0) = \lim_{t \to t_0} \frac{P_{t_0 t}^{-1} V(t) - V_0}{t - t_0}$. □

5.3 Riemannian Geodesics

Definition 5.23. *$\nabla^\top : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ by setting $\nabla^\top_X Y = \pi^\top(\overline{\nabla}_X Y)$ where X and Y are extended arbitrary to $\mathbb{R}^n$, $\overline{\nabla}$ is the Euclidean connection on $\mathbb{R}^n$ and for any point $p \in M$, $\pi^\top : T_p\mathbb{R}^n \to T_pM$ is the orthogonal projection.*

This is called the tangentiel connection on M.

Proposition 5.24. *Any vector field on M can be extended to a smooth vector field on $\mathbb{R}^n$*

Definition 5.25. *A linear connection ∇ is said to be compatible with g (a Riemann metric) if*

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle = \langle Y, \nabla_X Z \rangle$$

with X,Y,Z vector fields

Lemma 5.26. *∇ a linear connection on a Riemannian manifold.*

1. *∇ is compatible with g*

2. $\nabla g = 0$

3. *If V, W are vector field along any curve γ,*

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$$

4. *If V, W are parallel vector fields along a curve γ, Then $\langle V, W\rangle$ is constant.*

The tangentiel connection on any embedded submanifold of $\mathbb{R}^n$ is compatible with the induced Riemannian metric.

Definition 5.27.

- *The torsion tensor is defined as a $\binom{2}{1}$-tensor field, $\tau : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ such that $\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ where $[X, Y] = X(Y(f)) - Y(X(f))$.*

- *A linear connection ∇ is said to be symmetric if its torsion vanishes identically, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$.*

Theorem 5.28 (Fundamental Lemma of Riemannian Geometry)**.** *Let (M,g) be a Riemannian manifold (or pseudo-Riemannian)*

There exists a unique linear connection ∇ on M that is compatible with g and symmetric.

Proof. The uniqueness is shown by computation :

$$\langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle = \nabla_X \langle Y, Z\rangle = X\langle Y, Z\rangle$$

And symmetry gives $\langle \nabla^1_X Y - \nabla^2_X Y, Z\rangle = 0, \ \forall X, Y, Z$
Then existence is given by $\langle \nabla_{\partial_i} \partial_j, \partial_l\rangle$ and $\nabla_{\partial_i} \partial_j = \Gamma^m_{ij} \partial_m$. Thus we can define

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} + \partial_l g_{ij})$$

And then we have symmetry and proof of compatibility with

$$g_{ij;k} = \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}$$

which can be prooved to be zero thus $\nabla g = 0$. □

Proposition 5.29. *From the previous proof we have :*

1. $\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - X\langle Y, [Y, X]\rangle + \langle X, [Z, Y]\rangle$

2. $\langle \nabla_{\partial_i}\partial_j, \partial_l\rangle = \frac{1}{2}(\partial_i\langle\partial_j,\partial_l\rangle + \partial_j\langle\partial_l,\partial_i\rangle - \partial_l\langle\partial_i,\partial_j\rangle)$; $g_{ij} = \langle\partial_i,\partial_j\rangle$; $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^m\partial_m$

3. $\Gamma_{ij}^m g_{ml} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{li} + \partial_l g_{ij})$; $g_{ml}g^{lk} = \delta_m^k$

4. $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl}+\partial_j g_{li}+\partial_l g_{ij})$ *This formula certainly defined a connection in each chart, and because* $\Gamma_{ij}^k = \Gamma_{ji}^k$, *the connection is symmetric if* $g_{ij;k} = \partial_k g_{ij} - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}$.

Lemma 5.30. *All Riemannian geodesics are constant speed curve i.e.* $|\dot\gamma(t)|$ *is independent of* t.

Proof. γ geodesic thus $\dot\gamma$ vector field such that $D_t\dot\gamma = 0$ thus $\dot\gamma$ is parallel to γ. Hence $\langle\dot\gamma, /dot\gamma\rangle = 0$ because $\frac{d}{dt}\langle V,W\rangle = \langle D_tV,W\rangle + \langle V, D_tW\rangle$. □

Proposition 5.31 (Naturalness of the Riemannian Connection)**.** *Suppose* $\varphi : (M,g) \to (\tilde M, \tilde g)$ *an isometry*

1. φ *takes the Riemannian connection* ∇ *of* g *to the Riemannian connection* $\overline{\nabla}$ *of* $\tilde g$ *in the sense that :*

$$\varphi_*(\nabla_X Y) = \overline{\nabla}_{\varphi_* X}(\varphi_* Y)$$

2. *If* γ *is a curve in* M *and* V *is a vector field along* γ, *then*

$$\varphi_* D_t V = \tilde D_t(\varphi_* V)$$

3. φ *takes geodesics to geodesics : if* γ *is the geodesics in* M *from* p *with initial velocity* V, $\varphi\circ\gamma$ *is the geodesics in* $\tilde M$ *with initial point* $\varphi(p)$ *and initial velocity* $\varphi_* V$.

6 The exponential map

Definition 6.1. $\mathcal{E} = \{V \in TM :\ \gamma_V$ *is defined on an interval countaining* $[0,1]\}$. *Exponential map : exp :* $\mathcal{E} \to M$ *such that exp*$(V) = \gamma_V(1)$. *The restricted exponential map exp*$_p$ *is the restriction to* $\mathcal{E}\cap T_pM = \mathcal{E}_p$.

Proposition 6.2. 1. $\mathcal{E}$ *is an open subset of TM countaining the zero section, and each set* $\mathcal{E}_p$ *is star-shaped with respect to 0.*

2. *For each* $V \in TP$, *the geodesic* γ_V *is given by* $\gamma_V(t) = exp(tV)$ *for all* t *such that either side in defined.*

3. The exponential map is smooth.

Lemma 6.3 (Rescaling lemma)**.** *For any $V \in TM$, $c, t \in \mathbb{R}$, $\gamma_{cV}(t) = \gamma_V(ct)$ whenever either side is defined*

Proposition 6.4 (Naturalness of the Exponential map)**.** *Suppose $\varphi : (M, g) \to (\tilde{M}, \tilde{g})$ is an isometry. Then, for any p in M, the following diagram commutes :*

$$\begin{tikzcd} T_pM \arrow[r, "\varphi_*"] \arrow[d, "exp_p"] & T_{\varphi_*(p)}\tilde{M} \arrow[d, "exp_{\varphi(p)}"] \\ M \arrow[r, "\varphi"'] & \tilde{M} \end{tikzcd}$$

6.1 Normal neighborhood and normal coordinates

Lemma 6.5 (Normal neighborhood lemma)**.** *For any $p \in M$ there is a neighborhood $\mathcal{V}$ of the origin in T_pM and $\mathcal{U}$ of p in M such that $exp_p : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism.*

Proof. Use of the inverse function theorem on Manifolds once we have $(exp_p)_*$ inversible at the point 0.

Since T_pM is a vector space there is a natural identification $T_0(T_pM) = T_pM$. Let $V \in T_pM,\ f \in \mathcal{C}^\infty(M)$, We can choose τ a curve in T_pM starting at 0 and with initial velocity V (for exemple $\tau(t) = tV$), and we can observe that

$$\dot{\tau} = \tau_*(\frac{d}{dt}) = V$$

Hence

$$\dot{\tau} f = \frac{d}{dt}|_{t=0} f \circ \tau(t) = Vf$$

And now

$$(exp_p)_* V = (exp_p)_* \tau_*(\frac{d}{dt}) = (exp_p \circ \tau)_*(\frac{d}{dt})$$

$$= \frac{d}{dt}|_{t=0} exp_p(tV) = \frac{d}{dt}|_{t=0} \gamma_V(t) = V$$

□

Definition 6.6. • *Any open neighborhood $\mathcal{U}$ of $p \in M$ that is a the diffeomorphic image under exp_p of a star-shaped open neighborhood of $O \in T_pM$ (as in the preceding lemma) is called a* ***normal neighborhood*** *of p.*

- *If $\epsilon > 0$ is such that exp_p is a diffeomorphism on the ball $B_\epsilon(0) \subset T_pM$ (where the radius of the ball is measured with respect to the norm defined by g) then the image $exp_p(B_\epsilon(0))$ is called a* ***geodesic ball*** *in M.*

- *Also if the closed ball $\overline{B}_\epsilon(0)$ is contained in an open set $\mathcal{V} \subset T_pM$ on which exp_p is a diffeomorphism, then $exp_p(\overline{B}_\epsilon(0))$ is called a* ***closed geodesic ball*** *and $exp_p(\partial\overline{B}_\epsilon(0))$ is called a* ***geodesic sphere****.*

- *If $\varphi = E^{-1} \circ exp_p^{-1} : \mathcal{U} \to \mathbb{R}^n$ is a coordinate chart with $\mathcal{U}$ normal neighborhood of p, and $E : \mathbb{R}^n \to T_pM$ such that $E(x^1, ..., x^n) = x^i E_i$. Any such coordinates are called (Riemannian)* ***normal coordinates*** *centered at p. There is a one to one correspondence between normal coordinates charts and orthogonal basis.*

- *In any normal coordinate charts centered at p, r define the radial distance function by*

$$r(x) = \sqrt{\sum_i (x^i)^2}$$

and the unit radial vector field $\frac{\partial}{\partial r}$ by

$$\frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i}$$

- *In Euclidean space $r(x)$ is the distance to the origin and $\frac{\partial}{\partial r}$ is the unit vector field tangent to straight lines through the origin.*

Proposition 6.7 (Properties of normal coordinates)**.** *Let $(\mathcal{U}, (x^i))$ be any normal coordinate chart centered at p*

1. *For any $V = V^i\partial_i \in T_pM$ the geodesic γ_V starting at p with initial velocity vector V is represented in normal coordinates by the radial line segment $\gamma V(t) = (tV^1, ..., tV^n)$ as long as γ_V stays within $\mathcal{U}$.*

2. *The coordinates of p are $(0, ..., 0)$*

3. *The components of the metric at p are $g_{ij} = \delta_{ij}$*

4. *Any Euclidean ball $\{x; r(x) < \epsilon\}$ contained in $\mathcal{U}$ is a geodesic ball in M.*

5. *At any point $q \in \mathcal{U} \backslash p$, $\frac{\partial}{\partial r}$ is the velocity vector of the unit speed geodesic from p to q, and therefore has unit length with respect to g.*

6. *The first partial derivative of g_{ij} and the Christoffel symbols vanish at p*

$$\forall k, \ \frac{\partial g_{ij}}{\partial x^k}(p) = 0$$

$$\Gamma^k_{ij}(p) = 0$$

Definition 6.8. *Geodesics starting at p and lying in a normal neighborhood of p are called **radial geodesics**.*

Proof. Elements of proof :
$V = V^i \partial_i$ and we can recall that $\varphi = E{-}1 \circ exp_p^{-1} : \mathcal{U} \to \mathbb{R}^n$ is a normal coordinate chart. We have

$$\varphi(\gamma_V(t)) = \varphi(exp_p(tV)) = \varphi(exp_p(tV^i \partial_i)) = E^{-1}(tV^i \partial_i) = (tV^1, ..., tV^n)$$

Thus $\gamma_V(t)$ can be written as $(tV^1, ..., tV^n)$ in normal coordinates. Hence $\varphi(p) = \varphi(\gamma_V(0)) = (0, ..., 0)$. □

Definition 6.9. *An open set $W \subset M$ is called **uniformly normal** if there exists some $\delta > 0$ such that W is contained in a geodesic ball around each of its points i.e. $\forall g \in W, \exists \delta > 0 : \ W \subset exp_q(B_\delta(0))$*

Lemma 6.10. *Given $p \in M$ and any neighborhood $\mathcal{U}$ of p, there exists a uniformly normal neighborhood W of p contained in $\mathcal{U}$.*

6.2 Geodesics of the model spaces

- Euclidean Space : The Euclidean geodesics are straight lines, and constant-coefficient vector fields are parallel

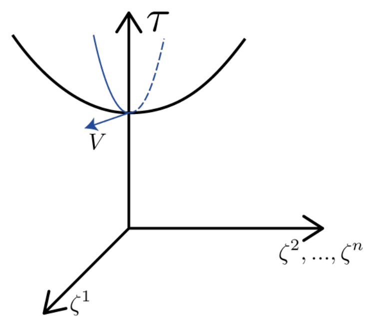
  Proof. □

- Spheres : The geodesics on S^n_R are precisely the "great circles" with constant speed parametrizations.

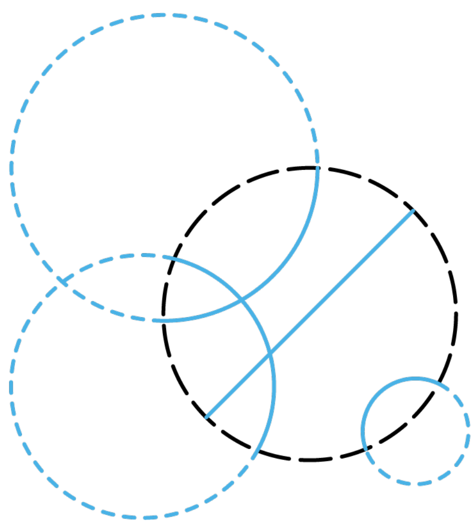
Proof. If we consider a geodesic starting from north pole N and initial velocity ∂_1 we can show that it is in the plane $x_2 = ... = x_n = 0$ in $\mathbb{R}^{n+1}$ because if $\exists i : \ x^i(t_0) \neq 0$ we can consider the isometry $\varphi(x^1, ..., x^{n+1}) = (x^1, ..., -x^i, ..., x^{n+1})$ but $N = \gamma(0)$ and $V = \dot{\gamma}(0)$ are invariant through φ, thus same geodesic, but $\varphi(\gamma(t_0)) \neq \gamma(t_0)$ contradiction. $\square$

- Hyperbolic spaces : The geodesics on the hyperbolic spaces are the following curves with constant speed parametrization

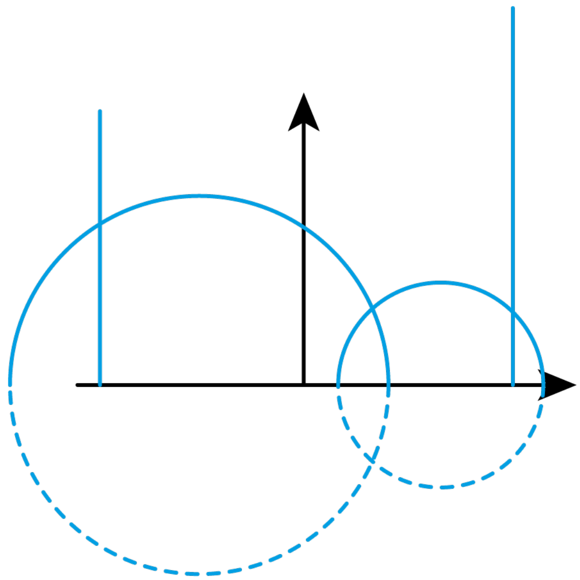
  1. Hyperboloid model : The "great hyperboles".



  2. Ball Model : The line segments through the origin and the circular arcs that intersects ∂B^n_R orthogonally.



  3. Half-space model : the vertical half lines and the semi-circles with centers on the $y = 0$ hyperplane.



7 Geodesics and distance

M is a smooth n-manifold endowed with a fixed Riemannian metric g

Definition 7.1. *If γ is a curve segment we define the length of γ to be*

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)|dt$$

Definition 7.2. *We define the reparametrization of γ to be a curve segment of the form $\tilde{\gamma} = \gamma \circ \varphi$ where $\varphi : [c, d] \to [a, b]$ is a smooth map with smooth inverse.*

Lemma 7.3. *For any curve segment $\gamma : [a, b] \to M$ and any reparametrization $\tilde{\gamma}$, $L(\gamma) = L(\tilde{\gamma})$.*

A regular curve is a smooth curve verifying $\forall t, \dot{\gamma}(t) \neq 0$ Geodesics are automatically regular, since they have constant speed.
A continuous map γ is called a piecewise regular curve segment if there exists a finite subdivision such that γ is a regular curve on each segment, they will be referred as **admissible curves**.

The length function of an admissible curve $\gamma : [a, b] \to M$ is the function $s : [a, b] \to \mathbb{R}$:

$$s(t) = L(\gamma|_{[a,b]}) = \int_a^t |\dot{\gamma}(u)|du$$

s is smooth where γ is and $\dot{s}(t) = |\dot{\gamma}(t)|$.

For each regular paramtrized $\mathcal{C}^r$-curve, where $r \geq 1$ the function s is defined and writing $\overline{\gamma}(s) = \gamma(t(s))$ where t is the inverse function of the function s (abuse of notation where s(t) is the opposite of t(s), s taking both as a variable and a function).

This is a reparametrization $\overline{\gamma}$ of γ called an arc-length parametrization, natural param., unit-speed param. and s(t) is called the natural parameter of γ, $|\dot{\overline{\gamma}}(s(t))| = 1$.

If γ is any admissible curve and $f \in \mathcal{C}^\infty([a,b])$ we define the integral of f with respect to arc length :

$$\int_\gamma f ds = \int_a^b f(t)|\dot{\gamma}(t)|dt$$

7.1 Riemannian distance function

Definition 7.4. *Suppose M is a connected Riemannian manifold. $\forall p, q \in M$ we can define the Riemannian distance $d(p,q)$ to be the infinimum of the lengths of all admissible curves from p to q.*

Lemma 7.5. *With the distance function d defined above any connected Riemannian manifold is a metric space whose induced topology is the same as the given manifold topology.*

7.2 Geodesics and minimizing curves

- An admissible curv γ in a Riemannian manifold is said to be minimizing if $L(\gamma) \leq L(\overline{\gamma})$ for any other admissible curve $\overline{\gamma}$ with the same endpoints.

- L is called a functionnal

- An admissible family of curves is a continuous map $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ that is smooth on each rectangle of the form $(-\epsilon, \epsilon) \times [-a_{i-1}, a_i]$ for some finite subdivision from a to b and such that $t \mapsto \Gamma_s(t) = \Gamma(s,t)$ is an admissible curve for each $s \in (-\epsilon, \epsilon)$.
  If Γ is an admissible family, a vector field along Γ is a continuous map $V : (-\epsilon, \epsilon) \times [a, b] \to TM$ such that $V(s,t) \in T_{\Gamma(s,t)}M$ for each (s,t) and such that $V|_{(-\epsilon,\epsilon)\times[\tilde{a}_{i-1},\tilde{a}_i]}$ is smooth for some subdivision $\tilde{a}_i$ from a to b.

- Any admissible family Γ defines two collections of curves : the main curves : $\Gamma_s(t) = \Gamma(s,t)$ defined on $[a,b]$ by setting s constant and the transverse curves $\Gamma^t(s) = \Gamma(s,t)$ defined on $(-\epsilon, \epsilon)$ by setting t constant. Transverse are smooth while main are only piecewise regular.

- Whenever Γ is smooth we denote :

$$\partial_t\Gamma(s,t) = \frac{d}{dt}\Gamma_s(t) \ ; \ \partial_s\Gamma(s,t) = \frac{d}{ds}\Gamma^t(s)$$

- If V is a vector field along Γ, we can compute the covariant derivation of V either along the main curves or along the transverse curves (at least where the former are smooth) resp. D_tV , D_sV.

Lemma 7.6. *Let Γ be an admissible family of curves in a Riemannian (or pseudo) manifold On any rectangle where Γ is smooth, we have*

$$D_s\partial_t\Gamma = D_t\partial_s\Gamma$$

Proof. A simple computation gives the identity. □

Definition 7.7. *If γ is an admissible curve, a variation of γ is an admissible family Γ such that $\Gamma_0(t) = \gamma(t)$, $\forall t \in [a,b]$. It is called a proper variation or fixed endpoint variation if in addition $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for any s.*

If Γ is a variation of γ, the variation field of Γ is the vector field $V(t) = \partial_s\Gamma(0,t)$ along γ (means that $V(t) \in T_{\gamma(t)}M$). A vector field V along γ is proper if $V(a) = V(b) = 0$.

Lemma 7.8. *If γ is an admissible curve and V a vector field along γ. Then V is the variation field of some variation of γ.*

Proof. Set $\Gamma(s,t) = exp(sV(t)) = \gamma_{V(t)}(s)$ thus $\partial_s\Gamma(s,t) = \dot\gamma_{V(t)}(s)$. But by definition $\dot\gamma_{V(t)}(O) = V(t)$ Thus $V(t) = \partial_s\Gamma(0,t)$ and V continuous on this whole domain. Conclusion V is a variation field of a variation of γ. □

Proposition 7.9 (First variation formula)**.** *Let γ be a unit speed admissible curve, Γ a proper variation of γ and V its variation field :*

$$\frac{d}{ds}|_{s=0}L(\Gamma_s) = -\int_a^b \langle V, D_t\dot\gamma\rangle dt - \sum_{i=1}^{k-1}\langle V(a_i), \triangle_i\dot\gamma\rangle$$

Where $\triangle_i\dot\gamma = \dot\gamma(a_i^+) - \dot\gamma(a_i^-)$ is the "jump" in the tangent vector field $\dot\gamma$ at a_i.

Proof. Since $L(\Gamma_s)$ is smooth and interval compact $[a_{i-1}, a_i]$ we can

$$\frac{d}{ds}L(\Gamma_s|_{[a_{i-1},a_i]}) = \frac{d}{ds}\int_{a_{i-1}}^{a_i} |\dot{\Gamma}(t)|dt = \int_{a_{i-1}}^{a_i} \frac{1}{|T|}\langle D_t S, T\rangle dt$$

with $T(s,t) = \partial_t\Gamma(s,t)$ and $S(s,t) = \partial_s\Gamma(s,t)$ by using that $\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$.

It is worth noting that $S(0,t) = V(t)$ as seen before and $T(0,t) = \dot{\gamma}(t)$ because Γ is proper. Hence by evaluating at $s = 0$ and γ being unit speed, we can use the relation of good behavior a second time and summing over i to get the identity.

Remark no need for unit speed curve because all curves have a unit speed reparametrization and the result is independent of parametrization. □

Theorem 7.10. *Every minimizing curve is a geodesic when it is given a unit speed parametrization.*

Proof. We have the right-hand side of identity of the previous proposition null for every V. So if we choose $V = \varphi D_t\dot{\gamma}$ with φ bump function on $]a_{i-1}, a_i[$ such that $V(a_i) = 0,\ \forall i \Rightarrow 0 = -\int_{a_{i-1}}^{a_i} \varphi|D_t\dot{\gamma}|^2 dt \Rightarrow D_t\dot{\gamma} = 0$. Then if we choose V a vector field along γ such that $V(a_i) = \triangle_i\dot{\gamma}$ and $V(a_j) = 0,\ \forall j \neq i$ we have $0 = -|\triangle_i\dot{\gamma}|^2$.
Thus the two one-sided velocity vectors of γ match up at each a_i, it follows from uniqueness of geodesics that $\gamma|_{[a_i,a_{i+1}]}$ is the continuation of the geodesic $\gamma|_{[a_{i-1},a_i]}$ and therefore smooth. □

Theorem 7.11 (The Gauss lemma)**.** *Let $\mathcal{U}$ be a geodesic ball centered at $p \in M$. The unit radial vector field $\frac{\partial}{\partial r}$ is g-orthogonal to the geodesic spheres in $\mathcal{U}$.*

Proof. □

Remark :

Let (x^i) be normal coordinates on a geodesic ball $\mathcal{C}$ centered at p in M and let the radial distance function as defined earlier :
Then $grad\ r = \frac{\partial}{\partial r}$ on $\mathcal{U}\backslash\{p\}$.

Proof. Since $\frac{\partial}{\partial r}$ is transverse to this sphere we can decompose Y as $\alpha\frac{\partial}{\partial r} + X$ for some constant α and some vector X tangent to the sphere. □

Proposition 7.12. *Suppose $p \in M$ and q is contained in a geodesic ball around p.*
Then (up to reparametrization) the radial geodesic from p to q is the unique minimizing curve from p to q in M.

Remark :

Within any geodesic ball around $p \in M$, the radial distance function $r(x)$ is equal to the Riemannian distance from p to q.
This gives that $\exp_p(B_R(0)) = B_R(p)$; $\exp_p(\overline{B}_R(0)) = \overline{B}_R(p)$ and $S_R(p) = exp_p(\partial B_R(0))$.

Definition 7.13. *We say a curve is locally minimizing if any $t_0 \in I$ has a neighborhood $\mathcal{U} \subset I$ such that $\gamma|_{\mathcal{U}}$ is minimizing between each pair of its point.*

Theorem 7.14. *Every Riemannian geodesic is locally minimizing.*

Proof. □

We can define the maximal geodesic $\gamma : I \to M$ naturaly. We have the property :
A Riemannian submanifold is to be geodesically complete if every maximal geodesic is defined for all $t \in \mathbb{R}$.

Theorem 7.15 (Hopf-Rinow)**.** *A connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space.*

8 Curvature

Local invariants :

- Non vanishing vector fields : In any suitable coordinates, every non vanishing vector field can be written locally as $V = \frac{\partial}{\partial x^1}$ so they are all locally equivalent.

- Riemannian metrics on a 1-manifold : If $\gamma : I \to M$ is a local unit speep parametrization of a Riemannian 1-manifold; then $s = \gamma^{-1}$ gives a coordinate chart in which the metric has the expression $g = ds^2$. Thus every Riemannian 1-manifold is locally isometric to $\mathbb{R}$

- 2-sphere and the Euclidean space are not locally isometric.

Definition 8.1. *If M is any Riemannian manifold the curvature endomorphism is the map $R : \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \times \to \mathcal{T}(M)$ defined by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Proposition 8.2. *The curvature endomorphism is a $\binom{3}{1}$-tensor field.*

Definition 8.3. *The Riemannian curvature tensor is defined as the covariant 4-tensor field :*

$$Rm = R^b$$

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$$

Lemma 8.4. *The Riemannian curvature endomorphism and curvature tensor are local isometry invariants. Formally : if $\varphi : (M,g) \to (\tilde{M},\tilde{g})$ is local isometry then $\varphi^*\tilde{Rm} = Rm$*

8.1 Flat manifold

Definition 8.5. *A Riemannian manifold is said to be flat if it is locally isometric to the Euclidean space that is, every point has a neighborhood that is isometric to an open set in $\mathbb{R}^n$ with its Euclidean metric.*

Theorem 8.6. *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proposition 8.7.

1. $R_{ijkl} = -R_{jikl}$
2. $R_{ijkl} = -R_{jilk}$
3. $R_{ijkl} = R_{ljik}$
4. $R_{ijkl} + R_{kijl} + R_{jkil} = 0$

Theorem 8.8 (Second Bianchi identity / Differiential Bianchi identity)**.** *The total covariant derivative of the curvature tensor satisfies the following identity :*

$$\nabla Rm(X,Y,Z,V,W) + \nabla Rm(X,Y,V,W,Z) + \nabla Rm(X,Y,W,Z,V) = 0$$

which can be written

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0$$

8.2 Ricci and scalar curvatures

Definition 8.9. *The component of the Ricci curvature are :*

$$R_{ij} = g^{km} R_{kijm}$$

And the scalar curvature is defined as the trace of the Ricci tensor :

$$S = tr_g Rc = R_i^i = g^{ij} R_{ij}$$

Lemma 8.10. *The Ricci curvature is a symmetric 2-tensor field.*

Lemma 8.11. *The covariant derivatives of the Ricci and scalar curvature satisfy the following identity*

$$divRc = \frac{1}{2}\nabla S$$

which can be written

$$R_{ij}{}^{;j} = \frac{1}{2}S_{;i}$$

Definition 8.12. *A Riemannian metric is said to be an Einstein metric if its Ricci tensor is a scalar multiple of the metric at each point i.e. for some function λ, $Rc = \lambda g$ everywhere. Taking trace on both side and noting that $tr_g g = g_{ij}g^{ij} = \delta_i^i = dim\ M$. We find that the Einstein condition can be written*

$$Rc = \frac{1}{n}Sg$$

Proposition 8.13. *If g is an Einstein metric on a connected manifold of dim $n \geq 3$ its scalar curvature is constant.*

8.3 Einstein field equation

The central assertion of Einstein's general theory of relativity is that physical space-time is modeled by a 4-manifold that carries a Lorentz metric whose Ricci curvature satisfies the following Einstein field equation :

$$Rc - \frac{1}{2}Sg = T \tag{3}$$

where T is a certain symmetric 2-tensor field (the stress energy tensor) that describe the density, momentum, and stress of the matter and energy present at each point in space time. (3) is the variational equation of a certain functional called the Hilbert action on the space of all Lorentz metrics on a given 4-manifold.

Einstein theory can be interpreted as the assertion that a physically realistic space time must be a critical point for this functional.

The vacuum Einstein field equation gives : $Rc = \frac{1}{2}Sg \Rightarrow S = 0 \Rightarrow Rc = 0$. Hence g is a Einstein metric in the mathematical sense of the word.

The studies in General Relativity generally are separate in two categories : The first being the studying of model of metrics that could verify the Einstein field equation in some particular cases, like the well-known Schwarzschild metric. While the second one is to study the geodesic in a given space. For exemple the light bending observed near a blackhole.

9 References

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